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Refraction of finite-amplitude water waves obliquely incident on a uniform beach

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The behaviour of a periodic wavetrain propagating obliquely over water of slowly varying depth is studied. The depth contours are taken to be straight and parallel. The wave properties used are those of 'numerically exact' solutions for waves on water of uniform depth. Comparison is made with linear theory which proves to be quite accurate for predicting wave direction unless the waves are propagating in a direction within about 25° of the contours. The results give a direct indication of where waves may break, but do not include dissipation.

Examples are given which correspond to waves 'trapped' within a region of limited depth. They are related to edge waves and to caustics of the linear theory. The behaviour of solutions is consistent with earlier work on deep-water waves. This includes behaviour we term 'anomalous refraction', which is to be discussed in another paper.

1. Introduction

The 'numerically exact' solutions for the integral properties of finite-amplitude periodic water waves propagating on water of infinite (Longuet-Higgins 1975) and finite (Cokelet 1977) depth have led to a number of papers which use them in 'slowly varying' situations. This paper gives solutions for waves obliquely incident on a beach which is uniform in the sense that its contours are straight and parallel. It builds on the paper by Stiassnie & Peregrine (1980) in which solutions are given for waves normally incident on a beach.

Appropriate equations for this problem are readily found from the books by Phillips (1977) and Whitham (1974). The two different approaches used by those authors are united and discussed by Stiassnie & Peregrine (1979). We take the opportunity to make some observations on them in § 2 where the mathematical methods used are described. It was found that some extension of the methods used by Stiassnie & Peregrine (1980) is required for waves at oblique incidence, but we follow them in using steady-wave solutions based on Cokelet's (1977) high-order Stokes-wave theory, and on a 'train of solitary waves' for long waves.

Section 3 of this paper gives some examples of the refraction of waves, incident on a beach from deep water, until they reach the minimum depth for which a solution can be found. At this limiting depth the waves are always steep, in the steepness range where many integral properties have their maximum values. It is reasonable to interpret this as an indication of breaking, though strictly it shows only that the approximate method of solution is invalid. There is discussion of this point in Stiassnie & Peregrine (1980).

In § 4 we proceed further and consider waves which cannot propagate to, or originate from, deep water. These are waves which are refracted in such a manner that they cannot travel beyond a certain maximum depth. Ray theory using infinitesimal waves gives a caustic at this critical depth, and waves propagating towards deeper water turn parallel with the shore at the caustic and then propagate towards shallower water as a reflected wave. An alternative way of considering such 'trapped' waves is as examples of high-order edge-wave modes. The solutions that are presented here give a clear indication that the caustics are similar to those on deep water (Peregrine & Thomas 1979; Peregrine 1981). Further, they are similar to the type-R caustics of Peregrine & Smith's (1979) classification of near-linear caustics.

One feature of all these caustics is that for a region near them the solution of the governing equations is not unique. There is a second steeper solution which has qualitatively different propagation properties. We use the term 'anomalous refraction' to describe their behaviour and discuss this and related topics in another paper (Peregrine and Ryrie, in preparation).

No dissipative terms are included. Garrett & Smith (1976) and Christoffersen & Jonsson (1980) show how these might be included in a wave-action equation. However, for steep waves there are no theoretical results and little experimental work from which to derive a reliable description of dissipation.

There are other recent papers describing waves incident on a uniform beach, using accurate wave solutions. Rienecker & Fenton (1981) illustrate their improvement of the 'stream-function method' for finding wave properties by considering waves at normal incidence on a beach. Their method is more accurate than Cokelet's for long waves, but does not describe waves close to the highest. Sakai & Battjes (1980) use Cokelet's results for this problem. However, they ignore the severe loss of accuracy of Cokelet's approach for long waves.

2. Method of solution

2.1. *The averaged equations*

A wave train propagating on water above a bottom at depth h^* below a reference level is defined by the mean water conditions, depth D^* and current $\mathbf{U}^*$, and wave properties, amplitude a^*, frequency ω^*, and wavenumber $\mathbf{k}^*$. The vectors are two-dimensional horizontal vectors, asterisks denote dimensional quantities, and all quantities are assumed to vary slowly with respect to Cartesian co-ordinates $\mathbf{x} = (x_1, x_2)$. The reference level for h^* is chosen to be the mean water level in deep water since this may best be regarded as an 'undisturbed water depth'. The reference frame for the wave quantities is chosen to be that in which $\mathbf{U}$ is the mean horizontal velocity at any point below the level of the wave troughs. This follows the convention of many previous authors, but does mean that there is a mass flow associated with the waves (that is when $\mathbf{U} = 0$).

As in Stiassnie & Peregrine (1979) general averaged equations may be written as follows:

$$\omega^* = k^*c^* + \mathbf{U}^*.\mathbf{k}^*, \tag{1}$$

$$\rho^*\frac{\partial D^*}{\partial t^*} + \nabla.(\rho^*D^*\mathbf{U}^* + \mathbf{I}^*) = 0, \tag{2}$$

$$\frac{\partial}{\partial t^*}\left(\frac{I^*}{k^*}\right) + \nabla.\left[\mathbf{U}^*\frac{I^*}{k^*} + (3T^* - 2V^* + \tfrac{1}{2}\rho^*D^*\overline{u_b^{*2}})\frac{\mathbf{k}^*}{k^{*2}}\right] = 0, \tag{3}$$

together with kinematic consistency relations

$$\frac{\partial k_1^*}{\partial x_2^*}-\frac{\partial k_2^*}{\partial x_1^*}=0, \quad \frac{\partial \mathbf{k}^*}{\partial t^*}+\nabla\omega^*=0, \tag{4), (5}$$

and further relations first given by Stiassnie & Peregrine (1979):

$$(\rho^* D^* \mathbf{U}^*+\mathbf{I}^*).\left(\frac{\partial \mathbf{U}^*}{\partial t^*}+\nabla\gamma^*\right)=0, \tag{6}$$

$$\frac{\partial U_\alpha^*}{\partial t^*}+\frac{\partial \gamma^*}{\partial x_\alpha^*}+\left(\frac{\partial U_\alpha^*}{\partial x_\beta^*}-\frac{\partial U_\beta^*}{\partial x_\alpha^*}\right)\left(U_\beta^*+\frac{I_\beta^*}{\rho^* D^*}\right)=0 \quad (\alpha,\beta=1,2). \tag{7}$$

In these expressions: c^* is the phase velocity, $\mathbf{I}^*$ is the average momentum density, T^* is the average kinetic-energy density, V^* is the average potential-energy density, $\overline{u_b^{*2}}$ is the mean-square wave-induced velocity at the bottom, and

$$\gamma^* = g^*(D^*-h^*)+\tfrac{1}{2}\mathbf{U}^{*2}+\tfrac{1}{2}\overline{u_b^{*2}}.$$

Equations (1)–(7) are an overdetermined set of equations. One of the superfluous equations is (6). It may be deduced by taking the scalar product of (7) with

$$\rho^* D^* \mathbf{U}^*+\mathbf{I}^*.$$

The second term becomes identically zero and the first leaves exactly (6).

Equations (6) and (7) were derived by Stiassnie & Peregrine (1979) to replace Whitham's (1974) consistency relations for pseudophase,

$$\frac{\partial \mathbf{U}^*}{\partial t^*}+\nabla\gamma=0, \tag{8}$$

$$\frac{\partial U_1^*}{\partial x_2^*}-\frac{\partial U_2^*}{\partial x_1^*}=0, \tag{9}$$

when the 'global' vorticity is zero. The curl of (8) reduces to

$$\frac{\partial}{\partial t^*}\left(\frac{\partial U_1^*}{\partial x_2^*}-\frac{\partial U_2^*}{\partial x_1^*}\right)=0,$$

Equation (9) should therefore be regarded as an initial condition on the vorticity, which remains zero if it is initially zero. Thus (8) is the only necessary consistency relation in that case. (This is briefly indicated by Hayes 1970.)

A similar argument applies to (4) and (5). The latter gives

$$\frac{\partial}{\partial t^*}\left(\frac{\partial k_1^*}{\partial x_2^*}-\frac{\partial k_2^*}{\partial x_1^*}\right)=0,$$

so (4) corresponds to an initial condition. These two cases can be compared with the use of the irrotationality condition and Kelvin's circulation theorem for inviscid flow.

2.2. *Periodic waves with unidirectional depth variation*

Attention is now restricted to monochromatic waves propagating over a bottom topography which has straight and parallel contours so that $h^* = h^*(x_1^*)$. Thus the above equations may be simplified by taking $\partial/\partial t^* = 0$ and $\partial/\partial x_2^* = 0$. By defining a local angle of incidence θ between $\mathbf{k}^*$ and the x_1^* direction, (4), (1), (2), (3) and (7) become, after some integrations,

$$k_2^* = k^* \sin\theta = \text{const.}, \tag{10}$$

$$\omega^* = c^* k^* + \mathbf{U}^* . \mathbf{k}^* = \text{const.}, \tag{11}$$

$$\rho^* D^* U_1^* + I_1^* = Q_1^* = \text{const.}, \tag{12}$$

$$\frac{U_1^* I_1^*}{k^*} + (3T^* - 2V^* + \tfrac{1}{2}\rho^* D^* \overline{u_b^{*2}}) \frac{k_1^*}{k^{*2}} = B_1^* = \text{const.}, \tag{13}$$

$$\frac{\partial \gamma^*}{\partial x_1^*} - \frac{\partial U_2^*}{\partial x_1^*}\left(U_2^* + \frac{I_2^*}{\rho^* D^*}\right) = 0, \tag{14}$$

$$\frac{\partial U_2^*}{\partial x_1^*}\left(U_1^* + \frac{I_1^*}{\rho^* D^*}\right) = 0. \tag{15}$$

In general, the vorticity $\partial U_2^*/\partial x_1^*$, is non-zero, and (14) and (15) are not readily integrable. However, as noted above, if the velocity field is initially irrotational it remains so. We consider a steady state that may eventually be attained when a regular wave train is incident onto a region in which the water is initially undisturbed. This implies that all effects due to wave breaking and dissipation are excluded. We include a constant value of U_2^* below, but as may be seen from (19) and (20) it may be neglected without loss of generality, and is considered to be equal to the velocity associated with the wave solutions in subsequent sections. The mass-flow component, Q_1, is set equal to zero as it would be for an impermeable beach.

All the variables in (10)–(15) are made dimensionless with the appropriate combinations of ρ^*, g^*, k^*. Note that this is a 'local' scaling since k^* is a function of position. The constants B_1^* and ω^*, and $\gamma^* = \text{const.}$, the integrated version of (14), are determined by reference to an 'initial state'. Typically this is deep water and it is convenient to use the value of k^* there to make these quantities dimensionless. This is denoted by k_∞^*.

After elimination of θ and U_1 by the use of (10) and (12) the set of equations becomes

$$\left(c - \frac{I}{D} + \frac{I}{D}\frac{m_2^2}{m^2}\right) m^{\frac{1}{2}} = 1 \tag{16}$$

$$\left(-\frac{I^2}{D} + 3T - 2V + \tfrac{1}{2}D\overline{u_b^2}\right)\left(1 - \frac{m_2^2}{m^2}\right)^{\frac{1}{2}} \frac{1}{m^3} = B_1 \omega_n^6 \tag{17}$$

$$\left[D - h + \tfrac{1}{2}\overline{u_b^2} + \tfrac{1}{2}I^2\left(1 - \frac{m_2^2}{m^2}\right)\Big/ D^2\right]\frac{1}{m} = \gamma_n \omega_n^2, \tag{18}$$

where

$$m = \frac{g^* k^*}{\omega_n^{*2}}, \quad m_2 = \frac{g^* k_2^*}{\omega_n^{*2}}$$

are dimensionless wavenumbers,

$$\omega_n = (\omega^* - U_2^* k_2^*)/(g^* k_\infty^*)^{\frac{1}{2}}, \tag{19}$$

$$\gamma_n = (\gamma^* - \tfrac{1}{2}U_2^{*2})\frac{k_\infty^*}{g^*}. \tag{20}$$

In practice it is the constants m_2, $B_1 \omega_n^6$ and $\gamma_n \omega_n^2$ which are determined at some reference point. These correspond respectively to longshore wavenumber, onshore wave-action flux and definition of reference level.

Once the constants and depth h are specified, (16) and (17) have as unknowns the wavenumber m and dimensionless wave properties. Two parameters are needed to specify these wave properties, for example Cokelet (1977) uses a steepness parameter ϵ^2 and a depth-related parameter d in his tabulation of wave properties.

2.3. *Solution methods*

To solve (16)–(18) it is easiest to treat h as an unknown to be found from (18). Then the wave parameter which appears in the equations in the most complicated manner is considered to be given. The approach of Stiassnie & Peregrine (1980) was followed in the solutions presented here. It needed some extension as indicated below.

For moderate to deep water, i.e. $k^*h^* \geqslant 0{\cdot}6$ a method based on Cokelet's (1977) high-order Stokes-wave solution was used. Wave properties are expanded as a power series in ϵ^2 with coefficients depending only on d; up to 50 terms are needed. Equation (16) is a quartic for m, and can be solved to eliminate m from (17). After much manipulation of the power series (17) can be expressed as the sum of powers of $\epsilon^{\frac{2}{3}}$, of the form

$$\sum_{n=0}^{N} a_n(d)\,\epsilon^{\frac{2}{3}n} = 0. \tag{21}$$

In Stiassnie & Peregrine (1980) the corresponding equation has a power series in ϵ^2 whose roots were found by a technique based on Padé approximants. This method proved impossible to use here due to numerical difficulties caused by the large differences in the magnitude of successive a_n. Accordingly the roots of (21) were found, for given d, by treating it simply as a polynomial. Often $N = 150$ and sometimes $N = 60$ sufficed. The accuracy of the results could be checked for the special case $\theta = 0$, by comparison with Stiassnie & Peregrine. Once ϵ^2 is found all other wave properties, depth and flow properties are readily calculated.

As observed by Stiassnie & Peregrine this method becomes inaccurate for $d \leqslant 0{\cdot}6$. In this work it is also found to be inaccurate for θ within about 10° of $\frac{1}{2}\pi$. An expansion for small $\phi = \frac{1}{2}\pi - \theta$ proved adequate, as described in § 4.

For values of $d \leqslant 0{\cdot}6$, that is for long waves, we follow Stiassnie & Peregrine in using a train-of-solitary-waves approximation (TSW). The two wave parameters for this approximation are $\mu = k^*h_s^*$, where h_s^* is the depth beneath the troughs and the height parameter ω_s of Longuet-Higgins & Fenton (1974). After some algebra (16) and (17) can be written

$$\frac{2\pi\mu^{\frac{1}{2}}F_s m^2}{2\pi+\mu M_s} - m^{\frac{3}{2}} + \mu^{\frac{3}{2}} m_2^2\left(\frac{M_s F_s}{2\pi+\mu M_s} - \frac{C_s}{2\pi}\right) = 0, \tag{22}$$

$$2\pi(m^2 - m_2^2)^{\frac{1}{2}} + m^4 B_1 \omega_n^6 (2\pi + \mu M_s) = 0, \tag{23}$$

where F_s, M_s and C_s are solitary-wave properties defined in Stiassnie & Peregrine (1980). As before, (22) is used to eliminate m, a value of ω_s is chosen and then (23) is solved numerically to find μ. As found by Stiassnie & Peregrine, the computation time is only 5–10 % of the time taken to use Cokelet's solution.

2.4. *Discussion of TSW approximation*

Since the TSW approximation proves to be rapid and convenient for calculating the integral properties of steep long waves, we have considered its possible applications a little further. In particular we have used the ninth-order solitary-wave solution of

Fenton (1979) to find the variation of horizontal velocity beneath the crests of waves as a function of height above the bed. On summing the resulting series and using Padé approximants the velocity profiles obtained were found to be 'reasonable' only for $\omega_s \leqslant 0{\cdot}75$. That is, the results were clearly incorrect for waves with heights greater than 0·69 times the depth. This is perhaps not surprising in view of Fenton's difficulty in finding satisfactory velocity profiles for fifth-order cnoidal waves with heights greater than 0·5 times the depth, and the discussion of Svendsen & Staub (1981) who conclude that the standard solitary- and cnoidal-wave theories should not be used to calculate velocity fields for high waves when greater than second-order accuracy is required.

It appears that the revised version of the stream-function method of calculating wave properties described by Rienecker & Fenton (1981) is more successful at describing velocity fields (A. New, private communication). It has the advantage that it is applicable over a greater range of wavelengths than is Cokelet's method. In this context, where we are exploring the character of large amplitude solutions it has the disadvantage that it does not accurately describe properties of waves close to the highest. As in Stiassnie & Peregrine (1980) we take the limiting high-wave solutions to indicate the proximity of breaking.

3. Shoaling waves

Sample solutions obtained by the methods described in §2 are used to illustrate their behaviour for waves propagating over a gently shoaling bottom from deep water. Since the equations are algebraic these solutions also hold for other depth variations such as deepening water or water with troughs and banks as long as the solutions exist at all intermediate points.

We use deep water as a reference point and indicate conditions in deep water by the subscript ∞. The constants m_2 and $B_1\omega_n^6$ are specified by the deep-water wave steepness $a_\infty^* k_\infty^*$, or equivalent, and direction θ_∞.

Eight different values of deep-water steepness were chosen and for each wave properties were calculated at six different angles of incidence. The steepnesses are:

$$a_\infty = a_\infty^*/k_\infty^* = 0{\cdot}307,\ 0{\cdot}197,\ 0{\cdot}131,\ 0{\cdot}0409,\ 0{\cdot}0314,\ 0{\cdot}0157,\ 0{\cdot}003\,14,\ 0{\cdot}001\,57.$$

These are the same as those given for $\theta = 0$ in Stiassnie & Peregrine (1980). The first four correspond to

$$\epsilon_\infty^2 = 0{\cdot}5,\ 0{\cdot}22,\ 0{\cdot}1,\ 0{\cdot}01,$$

the remainder to

$$H_\infty/L_\infty = 0{\cdot}01,\ 0{\cdot}005,\ 0{\cdot}001,\ 0{\cdot}0005,$$

where $H = 2a$ is wave height and L is wavelength.

Selected results are presented in the figures. Figure 1 shows the variation of amplitude $a = a^*k_\infty^*$ with depth $h = h^*k_\infty^*$, for each value of a_∞, for $\theta_\infty = 0$ and $\theta_\infty = 60°$. The results for $\theta_\infty = 0$ agree with those of Stiassnie & Peregrine (1980). For $\theta_\infty = 60°$ the results predicted by linear theory are also shown, together with the value of θ at the least depth with a solution.

Figure 2 shows the variation of amplitude with depth for one initial steepness $a_\infty = 0{\cdot}13$, at five angles of incidence. It includes examples of anomalous refraction, which for this initial steepness only occur when $83° < \theta_\infty < 90°$. Similarly, figure 3,

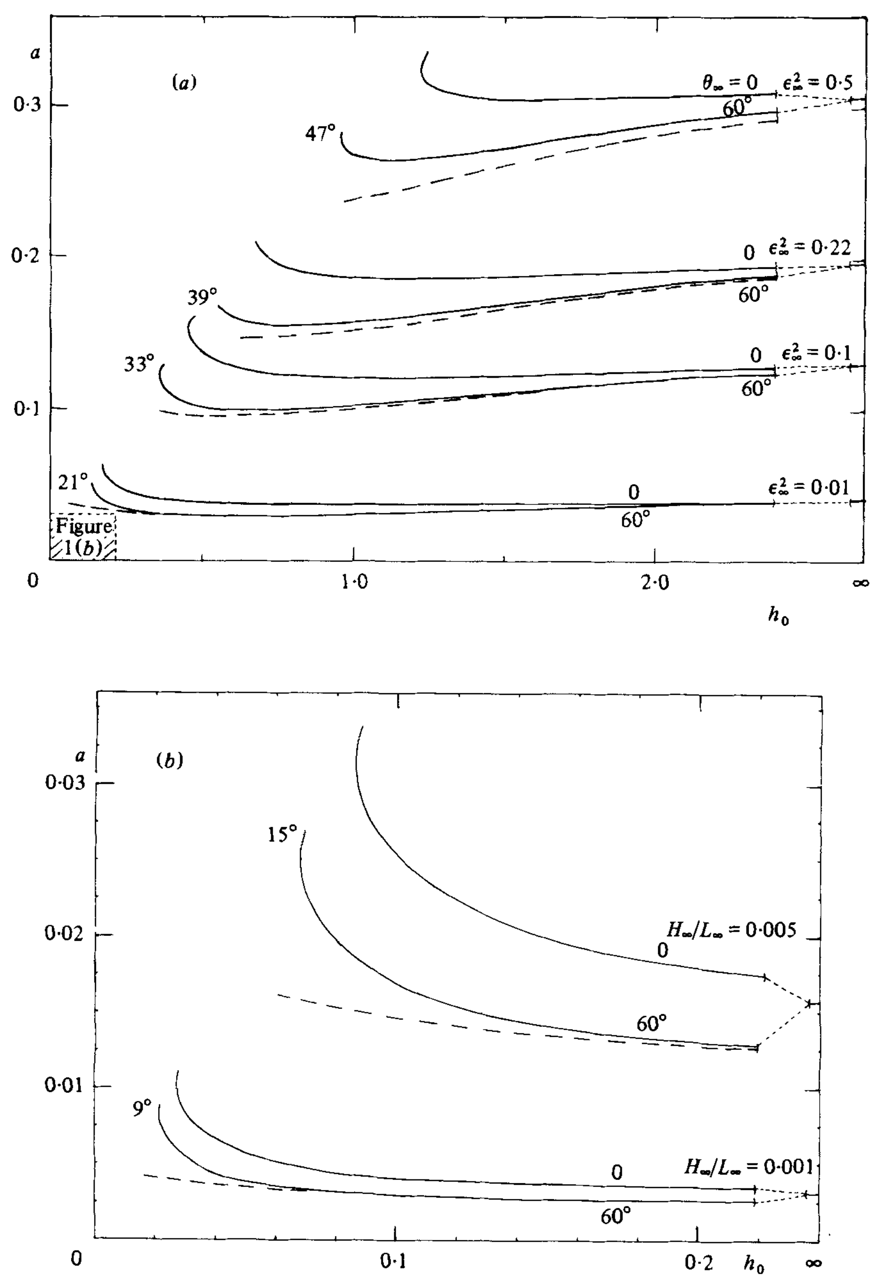

FIGURE 1. Variation of wave amplitude with undisturbed depth for $\theta_\infty = 0$ and 60°. The broken line shows linear theory for $\theta_\infty = 60°$. Values of angle of incidence at 'breaking', θ_b, are shown.

which shows the variation of θ with depth for many of the calculated cases, includes anomalous refraction; the lines for $\epsilon_\infty^2 = 0·1$ and $0·5$ at $\theta_\infty = 85°$. Anomalous refraction is discussed by Peregrine & Ryrie (in preparation).

As might be expected linear theory gives a fair guide to wave refraction for gentle waves. The deviation from linear theory as waves steepen is more marked in wave amplitude than direction. The linear solutions are not marked in figure 3 since they

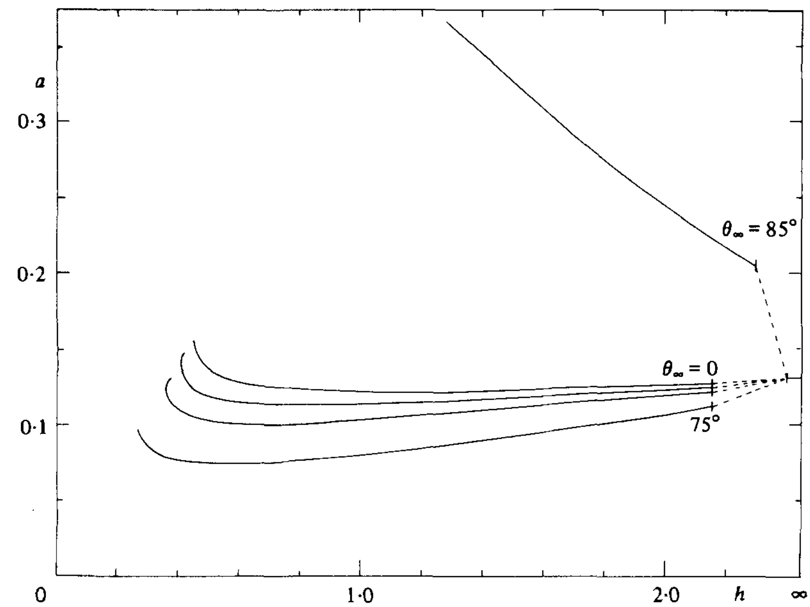

FIGURE 2. Variation of wave amplitude with undisturbed depth for $\epsilon_\infty^2 = 0{\cdot}1$, with $\theta_\infty = 0$, 45°, 60°, 75°, 85°.

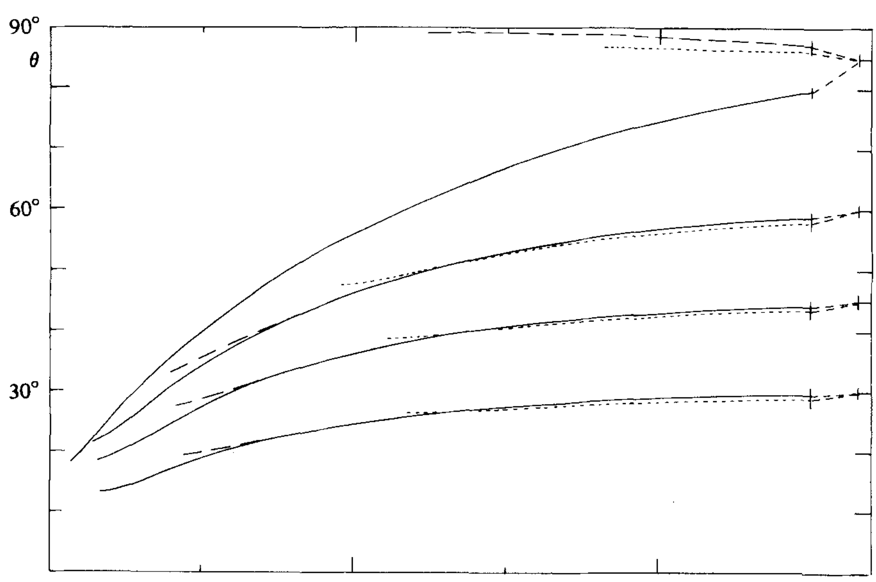

FIGURE 3. Variation of angle of incidence θ with depth h.
——, $\epsilon_\infty^2 = 0{\cdot}01$; – – – –, $\epsilon_\infty^2 = 0{\cdot}1$; - - - -, $\epsilon_\infty^2 = 0{\cdot}5$.

are very close to the line for $\epsilon_\infty^2 = 0{\cdot}01$ except at its shallow-water extremity. All solution curves are limited in their depth range. As h decreases a depth is reached at which the solution has a singular gradient and our slowly varying approximation becomes invalid. This occurs at a wave steepness corresponding to the region of maxima in wave integral properties. There is also a second very steep solution for slightly

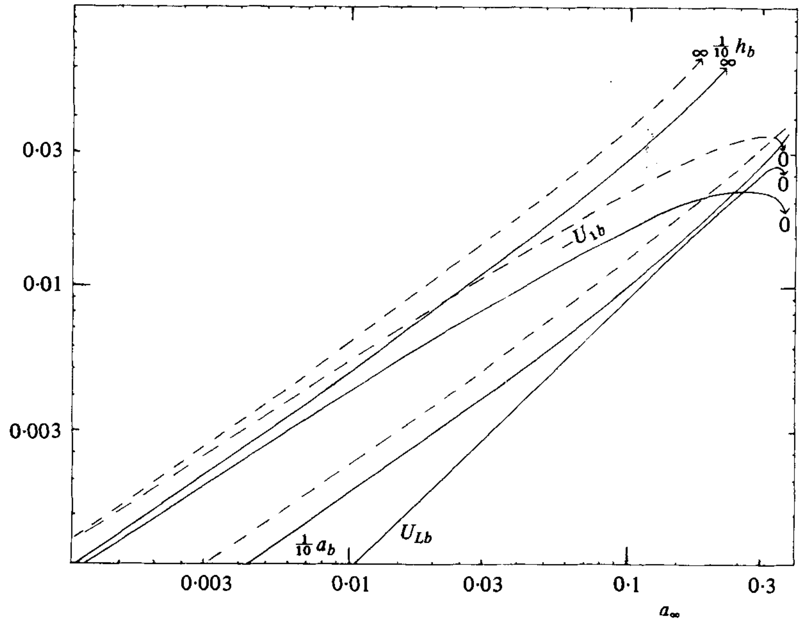

FIGURE 4. Variation of depth h_b, amplitude a_b, onshore current U_b, and longshore Stokes-drift velocity U_{Lb}, evaluated at 'breaking' point, with deep-water steepness $a_\infty = a_\infty^* k_\infty^*$. ——, $\theta_\infty = 60°$; - - -, $\theta_\infty = 0$.

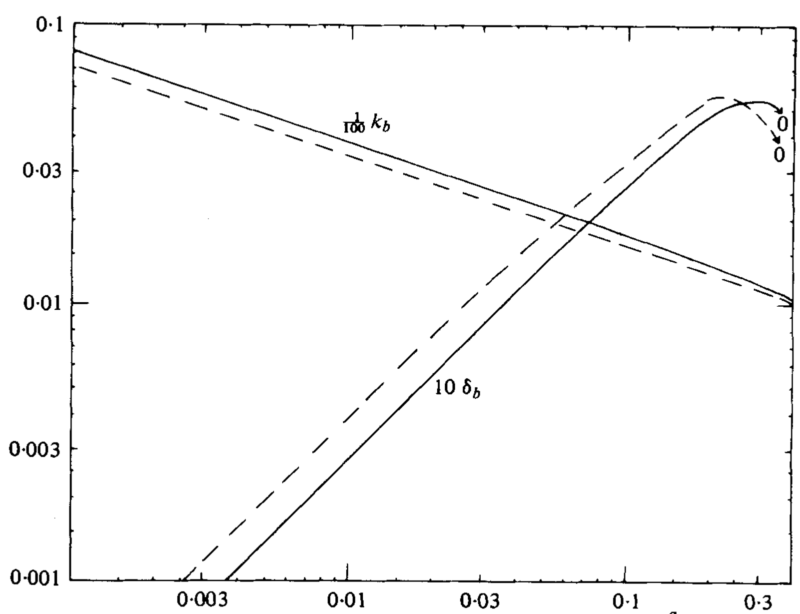

FIGURE 5. Variation of set-down δ_b and wavenumber k_b, evaluated at the 'breaking' point, with deep-water steepness $a_\infty = a_\infty^* k_\infty^*$. ——, $\theta_\infty = 60°$; - - -, $\theta_\infty = 0$.

greater depths. However, that second solution is very likely to be unstable. For these reasons, which are discussed in detail in Stiassnie & Peregrine (1980), it is reasonable to assume that wave-breaking occurs close to the singular depth.

Once the primary wave parameters have been found by the methods of § 2 all other wave properties are readily found. Figures 4 and 5 illustrate a number of these at the 'breaking point', that is, at the singular depth discussed above, as functions of a_∞. Once again the two cases $\theta_\infty = 0$ and $\theta_\infty = 60°$ are shown. The graphs are limited to $a_\infty < 0{\cdot}41$, the value above which rapidly growing instabilities occur (Longuet-Higgins, 1978).

All the various properties shown in figures 4 and 5 have a difference between the values for $\theta_\infty = 0$ and $\theta_\infty = 60°$ which is largely due to the shallower depth to which the latter waves propagate before breaking. This shallower depth may be interpreted as due to the smaller energy flux incident per unit length of beach.

4. Trapped waves

Most refraction problems for water waves assume implicitly that waves approach shallow water from deep water. However, waves do propagate from shallow water toward deeper water; for example, after propagating over a bar, or when approaching a dredged channel or submarine canyon. In these cases linear theory predicts that waves at a sufficiently large angle of incidence are reflected from a caustic region beyond which these waves do not propagate. More precisely for the analysis we have considered in § 2 waves may form a caustic if $m_2 > 1$. Solutions corresponding to such waves have no equivalent deep-water wave (except that in linear theory the wave-number k_1 may be complex) and hence are not included in § 3.

If waves are reflected at the shoreline and at a caustic, they are trapped; in the linear theory they are described by edge-wave modes. In this work the beach is assumed to be so gentle that breaking rather than reflection is expected, and as a result our assumption of one wavetrain is somewhat more realistic. However, the neglect of interactions with waves reflected from a caustic is an important omission which is being studied further. Nonetheless the results are still of interest.

The simplest of these waves is one which only propagates parallel to the depth contours. Once the wavenumber is chosen there is a single set of solutions for which phase velocity is an appropriate parameter. The phase velocity determines the depth at which an infinitesimal plane wave can travel parallel to the shore and also determines the least depth at which a finite-amplitude wave travels parallel to the shore. The finite-amplitude portion of the wave, in shallower water than the zero-amplitude limit, can travel at the same speed by virtue of its greater steepness. These solutions have $B_1 = 0$, and examples are included in figures 6 and 7.

For other solutions there is now no special reference point, such as deep water, for specifying initial conditions, so we simply choose values of $B_1 \omega_n^6$ and m_2 and solve (16)–(18) using the methods of § 2. We take $\gamma_n \omega_n^2 = 0$ as before. The value of m_2 determines the position of any linear caustic and $B_1 \omega_n^6$, the wave action, is a measure of the intensity.

The calculated solutions appear to be satisfactory, except when θ is near $\frac{1}{2}\pi$. However, the case $\theta = \frac{1}{2}\pi$ is the simple example mentioned above. Thus by expanding in

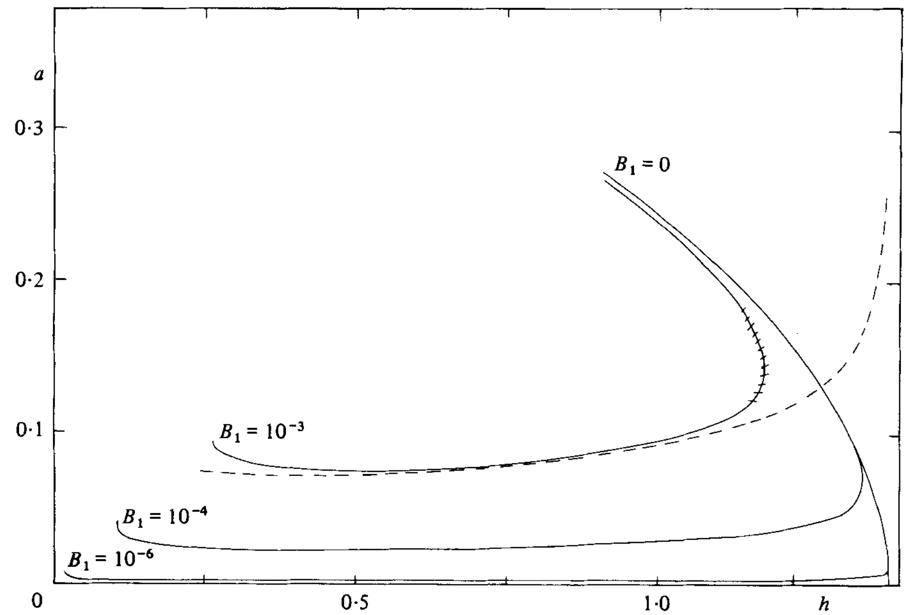

FIGURE 6. Variation of amplitude with undisturbed depth, for trapped waves, with longshore wavenumber $m_2 = 1{\cdot}1$, at different values of onshore component of wave-action flux B_1. The singularity in the linear solution occurs at $h^*\omega^{*2}/g^* = 1{\cdot}38$. The broken line shows the linear solution for $B_1 = 10^{-3}$. The solution in the hatched region is less well determined than elsewhere.

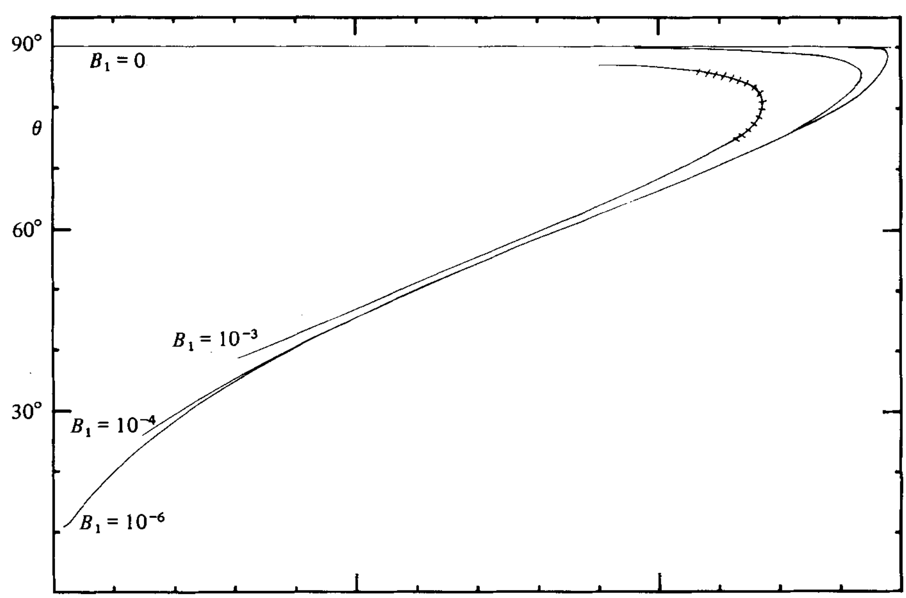

FIGURE 7. Variation of angle of incidence θ with undisturbed depth for trapped waves, with longshore wavenumber $m_2 = 1{\cdot}1$. The solution in the hatched region is less well determined than those elsewhere.

terms of $\phi = \frac{1}{2}\pi - \theta$ it is a straightforward calculation to approximate (16) and (17) by

$$\phi^2 - \frac{2D(1-m_2c^2)}{d+\bar{\eta}(1-4m_2c^2)} = O(\phi^4), \tag{24}$$

$$(-c^2\bar{\eta}^2/D + 3T - 2V + \tfrac{1}{2}D\overline{u_b^2})\,\phi - B_1\omega_n^6 m_2^3 = O(\phi^3), \tag{25}$$

where $\bar{\eta} = D-d$.

Results are given for the case

$$m_2 = g^*k_2^*/\omega_n^{*2} = 1{\cdot}1,$$

for which the linear caustic is at

$$h = h^*\omega_n^{*2}/g^* = 1{\cdot}384,$$

for values of wave-action flux given by

$$B_1\omega_n^6 = 0,\ 10^{-6},\ 10^{-4},\ 10^{-3}.$$

Figures 6 and 7 show the variation of amplitude and angle of incidence with depth for these cases. The perturbation solution described above and the methods of § 2 are adequate for all except the most intense waves considered ($B_1\omega_n^6 = 10^{-3}$) for which a region of doubtful accuracy remains. Taking further terms into account in (24) and (25) can be expected to improve accuracy here but the general trend of the solution is sufficiently clear.

Near the caustic there are two solutions. The first lies close to the linear solution but reaches a singularity, where the solution curve has a vertical tangent, just inshore of the linear caustic position. Its amplitude increases very little as the singularity is approached. Indeed for the case of $B_1 = 0$ the singularity is at zero amplitude. The pattern of the solution corresponds to the type-R near-linear caustic of Peregrine & Smith (1979) and is unlikely to be associated with wave-breaking. As indicated there, consideration of higher-order modulation effects gives a smooth reflecting solution. The reflected wave-train simply has the sign of B_1 reversed and the solution properties, other than those associated with the propagation direction, are unchanged.

In this case we do not include the interaction between incident and reflected waves, as already mentioned. The solution needed to deal with this region is thus not fully elucidated. However, for the case $B_1 = 0$, the linear approximation may be sufficient since the amplitude is zero at the singularity. An appropriate solution is part of a high-modenumber edge-wave solution with the same phase velocity.

The solutions all have a second singularity, of the type we associate with breaking, in shallower water. There is also a second branch to the solution curves for non-zero B_1. This branch is close to the $B_1 = 0$ solution and exhibits anomalous refraction. Once again, all other wave properties can be calculated. In figure 8, the set down δ and longshore mean drift velocity, U_L, are given for the cases $B_1 = 0$ and 10^{-4}. As may be seen, these are always a small fraction of the appropriate parameters, e.g. $-\delta/h < 0{\cdot}006$, $U_2/c < 0{\cdot}04$. This is of relevance to the interpretation of results. The fact that the waves influence the propagating medium by modifying the depth and currents means that the analysis of Peregrine & Smith (1979) is not strictly applicable since these are not taken into account. However, since the calculated results show that the modifications of the medium are small, one can expect that these results,

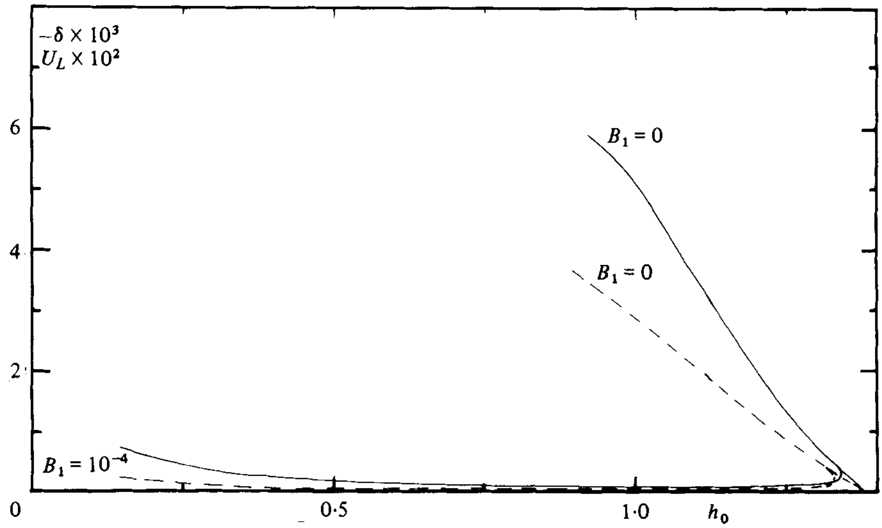

FIGURE 8. Variation of set-down δ and mean longshore Stokes-drift velocity U_L for waves near a caustic, with $m_2 = 1{\cdot}1$, $B_1 = 0$ and 10^{-4}. —— δ, U_L – – – –.

and others that may be derived in future, for the simpler 'wave-only' case are only slightly modified for application in this case.

5. Conclusion

This work was commenced with the impression that it would be a straightforward extension of Stiassnie & Peregrine (1980). The analysis and computation proved to be more complicated than expected. The differences between these finite-amplitude solutions and linear theory is no greater than might be expected; indeed it appears that wave direction is little different from linear theory for waves heading within 60° of the line of greatest beach slope.

Big differences from other theories occur in two circumstances. For the steepest waves, the limiting solutions give a clear indication that breaking may occur, and the variation of properties for slightly less steep waves is markedly different from linear theory. However it is similar to the results of Stiassnie & Peregrine.

The other large differences are for waves which propagate almost perpendicularly to the direction of greatest depth variation. In this case, waves can be refracted in a sense which is opposite to that of linear waves; hence we introduce the term anomalous refraction. It is not entirely unexpected since studies of waves near caustics (Peregrine & Smith 1979; Peregrine & Thomas 1979; Peregrine 1981) show similar behaviour. The matter is discussed in more detail in Peregrine & Ryrie (in preparation), where it is noted that in the limit as the angle between wave propagation direction and the depth gradient approaches 90°, the linear approximation is qualitatively correct only for zero-amplitude waves.

The methods we use for calculating properties of the steady wave are unlikely to be the most convenient in practice; they involve the use of two separate methods, for long and for short waves, in order to obtain accurate solutions for all wave steepnesses. A single computation method such as Rienecker & Fenton's (1981) recent improvement of the 'stream-function method', which covers a full range of depths, but not the steepest waves, may be better, since where wave steepness is near maximum, wave properties vary rapidly, and the slowly varying assumption is not valid; our solutions, which do not take into account any interaction with bottom slope, may not be accurate here.

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